2004

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IJMMS 2004:11, 579–598
PII. S0161171204305260
http://ijmms.hindawi.com

INTEGRAL TRANSFORMS, CONVOLUTION PRODUCTS, AND FIRST VARIATIONS

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Received 21 May 2003

We establish the various relationships that exist among the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$, and the first variation δF for a class of functionals defined on $K[0,T]$, the space of complex-valued continuous functions on $[0,T]$ which vanish at zero.

2000 Mathematics Subject Classification: 28C20.

1. Introduction and definitions. In a unifying paper [10], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform [2], the Fourier-Feynman transform [3], and the Gauss transform are special cases of his integral transform $\mathcal{F}_{\alpha,\beta}$. In [5], Chang et al. established an interesting relationship between the integral transform and the convolution product for functionals on an abstract Wiener space. In this paper, we study the relationships that exist among the integral transform, the convolution product, and the first variation [1, 4, 9, 11].

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. Then $(C_0[0,T],\mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)m(dx). \tag{1.1}$$

Let $K=K[0,T]$ be the space of complex-valued continuous functions defined on $[0,T]$ which vanish at $t=0$. Let α and β be nonzero complex numbers. Next we state the definitions of the integral transform $\mathcal{F}_{\alpha,\beta}F$, the convolution product $(F*G)_\alpha$, and the first variation δF for functionals defined on K.

Definition 1.1. Let F be a functional defined on K. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}(F)(y)\equiv\mathcal{F}_{\alpha,\beta}F(y)\equiv\int_{C_0[0,T]} F(\alpha x+\beta y)m(dx),\quad y\in K, \tag{1.2}$$

if it exists [5, 8, 10].

Definition 1.2. Let F and G be functionals defined on K. Then the convolution product $(F*G)_\alpha$ of F and G is defined by

$$(F*G)_\alpha(y) \equiv \int_{C_0[0,T]} F\left(\frac{y+\alpha x}{\sqrt{2}}\right) G\left(\frac{y-\alpha x}{\sqrt{2}}\right) m(dx), \quad y \in K, \tag{1.3}$$

if it exists [5, 7, 13, 14].

Definition 1.3. Let F be a functional defined on K and let $w \in K$. Then the first variation δF of F is defined by

$$\delta F(y|w) \equiv \frac{\partial}{\partial t} F(y+tw)|_{t=0}, \quad y \in K, \tag{1.4}$$

if it exists [1, 4, 11].

Let $\{\theta_1, \theta_2, \dots\}$ be a complete orthonormal set of real-valued functions in $L_2[0,T]$. Furthermore, assume that each θ_j is of bounded variation on $[0,T]$. Also let $\operatorname{Var}(\theta_j,[0,T])$ denote the total variation of θ_j on $[0,T]$. Then for each $y \in K$ and $j \in \{1,2,\dots\}$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_0^T \theta_j(t)\,dy(t)$ exists. Furthermore,

$$|\langle \theta_j, y\rangle| = \left| \theta_j(T)y(T) - \int_0^T y(t)\,d\theta_j(t) \right| \le C_j \|y\|_\infty \tag{1.5}$$

with

$$C_j = |\theta_j(T)| + \operatorname{Var}(\theta_j,[0,T]). \tag{1.6}$$

Next we describe the class of functionals that we work with in this paper. Let E_0 be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(y) = f(\langle \theta_1, y\rangle, \dots, \langle \theta_n, y\rangle) \tag{1.7}$$

for some positive integer n, where $f(\lambda_1, \dots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \dots, \lambda_n$ of exponential type; that is to say,

$$|f(\lambda_1, \dots, \lambda_n)| \le A_F \exp\left\{ B_F \sum_{j=1}^n |\lambda_j| \right\} \tag{1.8}$$

for some positive constants A_F and B_F.

To simplify the expressions, we use the following notations. For $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, we write

$$\begin{gathered} \|\vec{u}\|^2 = \sum_{j=1}^n u_j^2, \qquad |\vec{u}| = \sum_{j=1}^n |u_j|, \qquad |\vec{\lambda}| = \sum_{j=1}^n |\lambda_j|, \qquad d\vec{u} = du_1 \cdots du_n, \\ f(\alpha\vec{u} + \beta\vec{\lambda}) = f(\alpha u_1 + \beta\lambda_1, \dots, \alpha u_n + \beta\lambda_n), \\ f(\langle \vec{\theta}, y\rangle) = f(\langle \theta_1, y\rangle, \dots, \langle \theta_n, y\rangle). \end{gathered} \tag{1.9}$$

Hence (1.7) and (1.8) can be expressed alternatively as

$$F(y) = f(\langle\vec{\theta}, y\rangle), \qquad |f(\vec{\lambda})| \le A_F \exp\{B_F|\vec{\lambda}|\}, \tag{1.10}$$

respectively. In addition, we use the notation

$$F_j(y) = f_j(\langle\vec{\theta}, y\rangle), \tag{1.11}$$

where $f_j(\vec{\lambda}) = (\partial/\partial\lambda_j) f(\lambda_1, \dots, \lambda_n)$ for $j = 1, \dots, n$.

In Section 2, we show that if F and G are elements of E_0, then $\mathcal{F}_{\alpha,\beta}F(\cdot)$, $(F * G)_\alpha(\cdot)$, $\delta F(\cdot|w)$, and $\delta F(y|\cdot)$ are also elements of E_0. In Section 3, we examine all relationships involving exactly two of the three concepts of "integral transform," "convolution product," and "first variation," while in Section 4, we examine all relationships involving all three of these concepts where each concept is used exactly once. For related work, see [2, 5, 7, 9, 10, 11, 13, 14] and for a detailed survey of previous work, see [12].

Remark 1.4. For any $F \in E_0$ and any $G \in E_0$, we can always express F by (1.7) and G by

$$G(x) = g(\langle\theta_1, x\rangle, \dots, \langle\theta_n, x\rangle) \tag{1.12}$$

using the same positive integer n, where f and g are entire functions of exponential type. For example, if $F \in E_0$ is of the form

$$F(x) = r(\langle\theta_1, x\rangle, \langle\theta_2, x\rangle), \tag{1.13}$$

and $G \in E_0$ is of the form

$$G(x) = s(\langle\theta_1, x\rangle, \langle\theta_3, x\rangle, \langle\theta_4, x\rangle), \tag{1.14}$$

where $r(\lambda_1, \lambda_2)$ and $s(\lambda_1, \lambda_3, \lambda_4)$ are entire functions of exponential type, then we can express F and G by (1.7) and (1.12) with $n = 4$ by choosing $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv r(\lambda_1, \lambda_2)$ and $g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv s(\lambda_1, \lambda_3, \lambda_4)$. In addition, the positive constants A_F, B_F, A_G, and B_G remain fixed. Thus throughout this paper, we will always assume that F and G belong to E_0 and are given by (1.7) and (1.12), respectively.

Remark 1.5. We considered various other classes of functionals before deciding to work exclusively with the class E_0 throughout this paper. One very natural class we considered was $L_2(C) \equiv L_2(C_0[0,T])$, the space of all complex-valued functionals F satisfying

$$\int_{C_0[0,T]} |F(x)|^2 m(dx) < \infty. \tag{1.15}$$

However in [8], Kim and Skoug showed that $L_2(C)$ is not invariant under the action of the integral transform operator. In fact, they showed that for every $\beta \in \mathbb{C}$ with $|\beta| > 1$, there exists a functional $F \in L_2(C)$ (the functional F depends on β) with $\mathcal{F}_{\alpha,\beta}(F) \notin L_2(C)$ even though $\alpha^2 + \beta^2 = 1$.

Another class of functionals we considered was

$$A = \{F \in L_2(C) : \mathcal{F}_{\alpha,\beta}(F) \in L_2(C)\ \forall \text{ nonzero } \alpha,\beta \in \mathbb{C}\}. \tag{1.16}$$

But for $F \in A$, the first variation δF of F may not exist; in fact, one needs some kind of a smoothness condition on F to even define δF.

As we will see in Section 2, E_0 is a very natural class of functionals in which to study the relationships that exist among the integral transform, the convolution product, and the first variation because for F and G in E_0, $\mathcal{F}_{\alpha,\beta}(F)$ and $(F * G)_\alpha$ exist and belong to E_0 for all nonzero complex numbers α and β, while $\delta F(y|w)$ exists and belongs to E_0 for all y and w in K. In addition, E_0 is a very rich class of functionals. Note that if E_0 is given by (1.7), then the entire function $f(\lambda_1,\dots,\lambda_n)$ is bounded if and only if it is a constant function. Thus many of the functionals in E_0 are unbounded, while for example, all of the functionals considered in [11] are bounded.

The so-called "tame functionals," that is, functionals of the form

$$G(x) = g(x(t_1),\dots,x(t_m)), \quad 0 < t_1 < \cdots < t_m \le T \tag{1.17}$$

as well as functionals of the form (1.7), played a major role in the development of Wiener space integration theory. But functionals of the form (1.17) are in E_0 provided $g(\lambda_1,\dots,\lambda_m)$ is an entire function of exponential growth. Included of course are all polynomials of m complex variables $\lambda_1,\dots,\lambda_m$ for all positive integers m, as well as such polynomials in $x(t_1),\dots,x(t_m)$ multiplied by functionals like $\exp\{\sum_{j=1}^m a_j x_j(t)\}$, and so forth.

2. The integral transform, the convolution product, and the first variation of functionals in E_0.

In our first theorem, we show that if F is an element of E_0, then the integral transform of F exists and is an element of E_0.

Theorem 2.1. *Let $F \in E_0$ be given by (1.7). Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ exists, belongs to E_0, and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(y) = h(\langle\vec{\theta},y\rangle) \tag{2.1}$$

for $y \in K$, where

$$h(\vec{\lambda}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\alpha\vec{u}+\beta\vec{\lambda})\exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\}d\vec{u}. \tag{2.2}$$

Proof. For each $y \in K$, using a well-known Wiener integration theorem, we obtain

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}F(y) &= \int_{C_0[0,T]} f(\alpha\langle\vec{\theta},x\rangle+\beta\langle\vec{\theta},y\rangle)m(dx)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\alpha\vec{u}+\beta\langle\vec{\theta},y\rangle)\exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\}d\vec{u}\\ &= h(\langle\vec{\theta},y\rangle),\end{aligned} \tag{2.3}$$

where h is given by (2.2). By [6, Theorem 3.15], $h(\vec{\lambda})$ is an entire function. Moreover, by inequality (1.8), we have

$$\begin{aligned} |h(\vec{\lambda})| &\le (2\pi)^{-n/2}\int_{\mathbb{R}^n} A_F \exp\Big\{B_F|\alpha\vec{u}+\beta\vec{\lambda}| - \frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u} \\ &\le A_{\mathcal{F}_{\alpha,\beta}F}\exp\{B_{\mathcal{F}_{\alpha,\beta}F}|\vec{\lambda}|\}, \end{aligned} \tag{2.4}$$

where

$$A_{\mathcal{F}_{\alpha,\beta}F} = A_F\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\Big\{-\frac{u^2}{2}+B_F|\alpha u|\Big\}du\right)^n < \infty \tag{2.5}$$

and $B_{\mathcal{F}_{\alpha,\beta}F} = B_F|\beta|$. Hence $\mathcal{F}_{\alpha,\beta}F \in E_0$. □

In our next theorem, we show that the convolution product of functionals from E_0 is an element of E_0.

Theorem 2.2. *Let $F, G \in E_0$ be given by (1.7) and (1.12) with corresponding entire functions f and g. Then the convolution $(F*G)_\alpha$ exists, belongs to E_0, and is given by the formula*

$$(F*G)_\alpha(y) = k(\langle\vec{\theta}, y\rangle) \tag{2.6}$$

for $y \in K$, where

$$k(\vec{\lambda}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f\left(\frac{\vec{\lambda}+\alpha\vec{u}}{\sqrt{2}}\right)g\left(\frac{\vec{\lambda}-\alpha\vec{u}}{\sqrt{2}}\right)\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u}. \tag{2.7}$$

Proof. For each $y \in K$, using a well-known Wiener integration theorem, we obtain

$$\begin{aligned} (F*G)_\alpha(y) &= \int_{C_0[0,T]} F\left(\frac{y+\alpha x}{\sqrt{2}}\right)G\left(\frac{y-\alpha x}{\sqrt{2}}\right)m(dx) \\ &= \int_{C_0[0,T]} f\left(\frac{\langle\vec{\theta},y\rangle+\alpha\langle\vec{\theta},x\rangle}{\sqrt{2}}\right)g\left(\frac{\langle\vec{\theta},y\rangle-\alpha\langle\vec{\theta},x\rangle}{\sqrt{2}}\right)m(dx) \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} f\left(\frac{\langle\vec{\theta},y\rangle+\alpha\vec{u}}{\sqrt{2}}\right)g\left(\frac{\langle\vec{\theta},y\rangle-\alpha\vec{u}}{\sqrt{2}}\right)\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u} \\ &= k(\langle\vec{\theta},y\rangle), \end{aligned} \tag{2.8}$$

where k is given by (2.7). By [6, Theorem 3.15], $k(\vec{\lambda})$ is an entire function and

$$\begin{aligned} |k(\vec{\lambda})| &\le (2\pi)^{-n/2}\int_{\mathbb{R}^n} A_F A_G \exp\Big\{\frac{B_F+B_G}{\sqrt{2}}(|\vec{\lambda}|+|\alpha||\vec{u}|) - \frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u} \\ &= A_{(F*G)_\alpha}\exp\{B_{(F*G)_\alpha}|\vec{\lambda}|\}, \end{aligned} \tag{2.9}$$

where $B_{(F*G)_\alpha} = (B_F+B_G)/\sqrt{2}$ and

$$A_{(F*G)_\alpha} = A_F A_G\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\Big\{-\frac{u^2}{2}+B_{(F*G)_\alpha}|\alpha u|\Big\}du\right)^n < \infty. \tag{2.10}$$

Hence $(F*G)_\alpha \in E_0$. □

In Theorem 2.3, we fix $w \in K$ and consider $\delta F(y|w)$ as a function of y, while in Theorem 2.4, we fix $y \in K$ and consider $\delta F(y|w)$ as a function of w.

Theorem 2.3. *Let $F \in E_0$ be given by (1.7) and let $w \in K$. Then*

$$\delta F(y|w) = p(\langle \vec{\theta}, y\rangle) \tag{2.11}$$

for $y \in K$, where

$$p(\vec{\lambda}) = \sum_{j=1}^{n} \langle \theta_j, w\rangle f_j(\vec{\lambda}). \tag{2.12}$$

Furthermore, as a function of $y \in K$, $\delta F(y|w)$ is an element of E_0.

Proof. For $y \in K$,

$$\begin{aligned} \delta F(y|w) &= \frac{\partial}{\partial t} f(\langle \vec{\theta}, y\rangle + t\langle \vec{\theta}, w\rangle)\big|_{t=0} \\ &= \sum_{j=1}^{n} \langle \theta_j, w\rangle f_j(\langle \vec{\theta}, y\rangle) = p(\langle \vec{\theta}, y\rangle), \end{aligned} \tag{2.13}$$

where p is given by (2.12). Since $f(\vec{\lambda})$ is an entire function, $f_j(\vec{\lambda})$ and so $p(\vec{\lambda})$ are entire functions. By the Cauchy integral formula, we have

$$f_j(\lambda_1,\ldots,\lambda_j,\ldots,\lambda_n) = \frac{1}{2\pi i}\int_{|\zeta-\lambda_j|=1} \frac{f(\lambda_1,\ldots,\zeta,\ldots,\lambda_n)}{(\zeta-\lambda_j)^2} d\zeta. \tag{2.14}$$

By inequality (1.8), for any ζ with $|\zeta - \lambda_j| = 1$, we have

$$\begin{aligned} \left|\frac{f(\lambda_1,\ldots,\zeta,\ldots,\lambda_n)}{(\zeta-\lambda_j)^2}\right| &\le A_F \exp\{B_F(|\lambda_1| + \cdots + |\zeta| + \cdots + |\lambda_n|)\} \\ &\le A_F \exp\{B_F|\vec{\lambda}| + B_F\}. \end{aligned} \tag{2.15}$$

Hence

$$|f_j(\vec{\lambda})| \le A_F e^{B_F} \exp\{B_F|\vec{\lambda}|\}, \tag{2.16}$$

and so

$$|p(\vec{\lambda})| \le \sum_{j=1}^{n} |\langle \theta_j, w\rangle| \, |f_j(\vec{\lambda})| \le A_{\delta F(\cdot|w)} \exp\{B_{\delta F(\cdot|w)}|\vec{\lambda}|\}, \tag{2.17}$$

where

$$A_{\delta F(\cdot|w)} = A_F e^{B_F} \|w\|_\infty \sum_{j=1}^{n} C_j < \infty \tag{2.18}$$

with C_j given by (1.6) and $B_{\delta F(\cdot|w)} = B_F$. □

Theorem 2.4. *Let $y \in K$ and let $F \in E_0$ be given by (1.7). Then*

$$\delta F(y|w) = q(\langle \vec{\theta}, w\rangle) \tag{2.19}$$

for $w \in K$, where

$$q(\vec{\lambda}) = \sum_{j=1}^{n} \lambda_j f_j(\langle \vec{\theta}, y\rangle). \tag{2.20}$$

Furthermore, as a function of w, $\delta F(y|w)$ is an element of E_0.

Proof. Equations (2.19) and (2.20) are immediate from the first part of the proof of Theorem 2.3. Clearly $q(\vec{\lambda})$ is an entire function. Next, using (2.16) we obtain

$$\begin{aligned} |q(\vec{\lambda})| &\le \sum_{j=1}^{n} |\lambda_j f_j(\langle \vec{\theta}, y\rangle)| \\ &\le A_F e^{B_F} \exp\{B_F[|\langle \theta_1, y\rangle| + \cdots + |\langle \theta_n, y\rangle|]\} \sum_{j=1}^{n} |\lambda_j| \\ &< A_F e^{B_F} \exp\{B_F \|y\|_\infty [C_1 + \cdots + C_n]\} e^{|\vec{\lambda}|} \\ &= A_{\delta F(y|\cdot)} \exp\{B_{\delta F(y|\cdot)} |\vec{\lambda}|\}, \end{aligned} \tag{2.21}$$

where $B_{\delta F(y|\cdot)} = 1$ and

$$A_{\delta F(y|\cdot)} = A_F e^{B_F} \exp\{B_F \|y\|_\infty [C_1 + \cdots + C_n]\}. \tag{2.22}$$

Hence, as a function of w, $\delta F(y|w) \in E_0$. □

We finish this section with some observations which we use later in this paper. First of all, (1.2) implies that

$$\mathcal{F}_{\alpha,\beta} F\left(\frac{y}{\sqrt{2}}\right) = \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y) \tag{2.23}$$

for all $y \in K$. Next, a direct calculation using (1.4), (1.2), (2.11), and (2.23) shows that

$$\delta \mathcal{F}_{\alpha,\beta} F\left(\frac{y}{\sqrt{2}} \middle| \frac{w}{\sqrt{2}}\right) = \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y|w) = \frac{\beta}{\sqrt{2}} \sum_{j=1}^{n} \langle \theta_j, w\rangle \mathcal{F}_{\alpha,\beta/\sqrt{2}} F_j(y) \tag{2.24}$$

for all y and w in K. Finally, by similar calculations, we obtain that

$$\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))\left(\frac{y}{\sqrt{2}}\right) = \frac{\sqrt{2}}{\beta} \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(y|w) \tag{2.25}$$

for all y and w in K, and for all $y \in K$,

$$(\mathcal{F}_{\alpha,\beta}F)_j(y) = \beta\mathcal{F}_{\alpha,\beta}(F_j)(y) = \beta\mathcal{F}_{\alpha,\beta}F_j(y). \tag{2.26}$$

3. Relationships involving two concepts. In this section, we establish all of the various relationships involving exactly two of the three concepts of integral transform, convolution product, and first variation for functionals belonging to E_0. The seven distinct relationships, as well as alternative expressions for some of them, are given by (3.1), (3.2), (3.4), (3.7), (3.9), (3.11), and (3.13).

In view of Theorem 2.1 through Theorem 2.4, all of the functionals that occur in this section are elements of E_0. For example, let F and G be any functionals in E_0. Then by Theorem 2.2, the functional $(F*G)_\alpha$ belongs to E_0, and hence by Theorem 2.1, the functional $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha$ also belongs to E_0. By similar arguments, all of the functionals that arise in (3.1) through (3.14) and (3.16) through (3.20) exist and belong to E_0.

Our first formula (3.1) is useful because it permits one to calculate $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha$ without ever actually calculating $(F*G)_\alpha$.

Formula 3.1. The integral transform of the convolution product of functionals from E_0 is given by the formula

$$\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y) = \mathcal{F}_{\alpha,\beta}F\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}G\left(\frac{y}{\sqrt{2}}\right) = \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y) \tag{3.1}$$

for all y in K.

Proof. Formula 3.1 is a special case of [5, Theorem 3.1]. □

Formula 3.2. The convolution product of the integral transform of functionals from E_0 is given by the formula

$$\begin{aligned}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y) &= (2\pi)^{-3n/2}\int_{\mathbb{R}^{3n}} f\left(\alpha\vec{r}+\frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle+\frac{\beta\alpha}{\sqrt{2}}\vec{u}\right)\\ &\quad\cdot g\left(\alpha\vec{s}+\frac{\beta}{\sqrt{2}}\langle\vec{\theta},y\rangle-\frac{\beta\alpha}{\sqrt{2}}\vec{u}\right)\exp\left\{-\frac{\|\vec{u}\|^2+\|\vec{r}\|^2+\|\vec{s}\|^2}{2}\right\}d\vec{u}\,d\vec{r}\,d\vec{s}\end{aligned} \tag{3.2}$$

for all y in K.

Proof. Using (1.3) and (1.2), we see that

$$\begin{aligned}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y) &= \int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}F\left(\frac{y+\alpha x}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}G\left(\frac{y-\alpha x}{\sqrt{2}}\right)m(dx)\\ &= \int_{C_0[0,T]}\left[\int_{C_0[0,T]}F\left(\alpha z_1+\frac{\beta(y+\alpha x)}{\sqrt{2}}\right)m(dz_1)\right]\end{aligned}$$

$$
\begin{aligned}
&\cdot \left[\int_{C_0[0,T]} G\left(\alpha z_2 + \frac{\beta(y-\alpha x)}{\sqrt{2}} \right) m(dz_2) \right] m(dx) \\
= &\int_{C_0^3[0,T]} f\left(\alpha\langle \vec{\theta}, z_1 \rangle + \frac{\beta}{\sqrt{2}} \langle \vec{\theta}, y \rangle + \frac{\alpha\beta}{\sqrt{2}} \langle \vec{\theta}, x \rangle \right) \\
&\cdot g\left(\alpha\langle \vec{\theta}, z_2 \rangle + \frac{\beta}{\sqrt{2}} \langle \vec{\theta}, y \rangle - \frac{\alpha\beta}{\sqrt{2}} \langle \vec{\theta}, x \rangle \right) m(dx) m(dz_1) m(dz_2).
\end{aligned} \tag{3.3}
$$

Formula (3.2) now follows upon evaluating the above Wiener integrals. □

Formula 3.3. The integral transform with respect to the first argument of the variation is given by the formula

$$
\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y) = \frac{1}{\beta}\delta \mathcal{F}_{\alpha,\beta} F(y|w) = \sum_{j=1}^{n} \langle \theta_j, w \rangle \mathcal{F}_{\alpha,\beta} F_j(y) \tag{3.4}
$$

for all y and w in K.

Proof. By applying Theorem 2.1 to expression (2.11), we obtain

$$
\begin{aligned}
\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(y) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} p(\alpha\vec{u} + \beta\langle \vec{\theta}, y \rangle) \exp\left\{ -\frac{1}{2}\|\vec{u}\|^2 \right\} d\vec{u} \\
&= (2\pi)^{-n/2} \sum_{j=1}^{n} \langle \theta_j, w \rangle \int_{\mathbb{R}^n} f_j(\alpha\vec{u} + \beta\langle \vec{\theta}, y \rangle) \exp\left\{ -\frac{1}{2}\|\vec{u}\|^2 \right\} d\vec{u}.
\end{aligned} \tag{3.5}
$$

On the other hand, by applying Theorem 2.3 to expression (2.1) and then using (2.2), we obtain

$$
\begin{aligned}
\frac{1}{\beta}\delta \mathcal{F}_{\alpha,\beta} F(y|w) &= \frac{1}{\beta} \sum_{j=1}^{n} \langle \theta_j, w \rangle h_j(\langle \vec{\theta}, y \rangle) \\
&= \frac{1}{\beta} \sum_{j=1}^{n} \langle \theta_j, w \rangle (2\pi)^{-n/2} \beta \int_{\mathbb{R}^n} f_j(\alpha\vec{u} + \beta\langle \vec{\theta}, y \rangle) \exp\left\{ -\frac{1}{2}\|\vec{u}\|^2 \right\} d\vec{u} \\
&= (2\pi)^{-n/2} \sum_{j=1}^{n} \langle \theta_j, w \rangle \int_{\mathbb{R}^n} f_j(\alpha\vec{u} + \beta\langle \vec{\theta}, y \rangle) \exp\left\{ -\frac{1}{2}\|\vec{u}\|^2 \right\} d\vec{u},
\end{aligned} \tag{3.6}
$$

and so (3.4) is established. □

Formula 3.4. The Integral transform with respect to the second argument of the variation is given by the formula

$$
\mathcal{F}_{\alpha,\beta}(\delta F(y|\cdot))(w) = \beta \delta F(y|w) \tag{3.7}
$$

for all y and w in K.

Proof. By applying Theorem 2.1 to expression (2.19), we obtain

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(\delta F(y|\cdot))(w) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} q(\alpha\vec{u}+\beta\langle\vec{\theta},w\rangle)\exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\}d\vec{u}\\ &= (2\pi)^{-n/2}\sum_{j=1}^{n}\int_{\mathbb{R}^n}(\alpha u_j+\beta\langle\theta_j,w\rangle)f_j(\langle\vec{\theta},y\rangle)\exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\}d\vec{u}\\ &= \beta\sum_{j=1}^{n}\langle\theta_j,w\rangle f_j(\langle\vec{\theta},y\rangle) = \beta\delta F(y|w).\end{aligned} \tag{3.8}$$

□

Formula 3.5. The first variation of the convolution product of functionals from E_0 is given by the formula

$$\delta(F*G)_\alpha(y|w) = \sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\left[(F_j*G)_\alpha(y)+(F*G_j)_\alpha(y)\right] \tag{3.9}$$

for all y and w in K.

Proof. By applying Theorem 2.3 to (2.6) and then using (2.7), we obtain

$$\begin{aligned}&\delta(F*G)_\alpha(y|w)\\ &\quad= \sum_{j=1}^{n}\langle\theta_j,w\rangle k_j(\langle\vec{\theta},y\rangle)\\ &\quad= (2\pi)^{-n/2}\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\int_{\mathbb{R}^n}\left[f_j\left(\frac{\langle\vec{\theta},y\rangle+\alpha\vec{u}}{\sqrt{2}}\right)g\left(\frac{\langle\vec{\theta},y\rangle-\alpha\vec{u}}{\sqrt{2}}\right)\right.\\ &\qquad\qquad\left.+f\left(\frac{\langle\vec{\theta},y\rangle+\alpha\vec{u}}{\sqrt{2}}\right)g_j\left(\frac{\langle\vec{\theta},y\rangle-\alpha\vec{u}}{\sqrt{2}}\right)\right]\exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\}d\vec{u}\\ &\quad= \sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle}{\sqrt{2}}\left[(F_j*G)_\alpha(y)+(F*G_j)_\alpha(y)\right].\end{aligned} \tag{3.10}$$

□

Formula 3.6. The convolution product, with respect to the first argument of the variation, of the variation of functionals from E_0 is given by the formula

$$(\delta F(\cdot|w)*\delta G(\cdot|w))_\alpha(y) = \sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle(F_j*G_l)_\alpha(y) \tag{3.11}$$

for all y and w in K.

Proof. Applying the additive distribution properties of the convolution product to the expressions

$$\delta F(y|w) = \sum_{j=1}^{n}\langle\theta_j,w\rangle F_j(y), \qquad \delta G(y|w) = \sum_{l=1}^{n}\langle\theta_l,w\rangle G_l(y) \tag{3.12}$$

yields (3.11) as desired. □

Formula 3.7. The convolution product, with respect to the second argument of the variation, of the variation of functionals from E_0 is given by the fromula

$$\big(\delta F(y|\cdot)*\delta G(y|\cdot)\big)_\alpha(w)=\frac{1}{2}\delta F(y|w)\delta G(y|w)-\frac{\alpha^2}{2}\sum_{j=1}^{n}F_j(y)G_j(y) \tag{3.13}$$

for all y and w in K.

Proof. Upon applying Theorem 2.2 to the expressions

$$\delta F(y|w)=\sum_{j=1}^{n}\langle\theta_j,w\rangle f_j(\langle\vec{\theta},y\rangle),\qquad \delta G(y|w)=\sum_{l=1}^{n}\langle\theta_l,w\rangle g_l(\langle\vec{\theta},w\rangle), \tag{3.14}$$

and using the fact that

$$\int_{\mathbb{R}^n}u_ju_l\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u}=\begin{cases}(2\pi)^{n/2} & \text{if } j=l,\\ 0 & \text{if } j\neq l,\end{cases} \tag{3.15}$$

we obtain

$$\begin{aligned}
&\big(\delta F(y|\cdot)*\delta G(y|\cdot)\big)_\alpha(w)\\
&\quad=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\Bigg[\sum_{j=1}^{n}\frac{\langle\theta_j,w\rangle+\alpha u_j}{\sqrt{2}}f_j(\langle\vec{\theta},y\rangle)\Bigg]\\
&\qquad\qquad\cdot\Bigg[\sum_{l=1}^{n}\frac{\langle\theta_l,w\rangle-\alpha u_l}{\sqrt{2}}g_l(\langle\vec{\theta},y\rangle)\Bigg]\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u}\\
&\quad=\frac{1}{2}(2\pi)^{-n/2}\sum_{j=1}^{n}\sum_{l=1}^{n}f_j(\langle\vec{\theta},y\rangle)g_l(\langle\vec{\theta},y\rangle)\\
&\qquad\qquad\cdot\int_{\mathbb{R}^n}(\langle\theta_j,w\rangle+\alpha u_j)(\langle\theta_l,w\rangle-\alpha u_l)\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2\Big\}d\vec{u}\\
&\quad=\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle f_j(\langle\vec{\theta},y\rangle)g_l(\langle\vec{\theta},y\rangle)-\frac{\alpha^2}{2}\sum_{j=1}^{n}f_j(\langle\vec{\theta},y\rangle)g_j(\langle\vec{\theta},y\rangle)\\
&\quad=\frac{1}{2}\Bigg[\sum_{j=1}^{n}\langle\theta_j,w\rangle f_j(\langle\vec{\theta},y\rangle)\Bigg]\Bigg[\sum_{l=1}^{n}\langle\theta_l,w\rangle g_l(\langle\vec{\theta},y\rangle)\Bigg]-\frac{\alpha^2}{2}\sum_{j=1}^{n}F_j(y)G_j(y)\\
&\quad=\frac{1}{2}\delta F(y|w)\delta G(y|w)-\frac{\alpha^2}{2}\sum_{j=1}^{n}F_j(y)G_j(y).
\end{aligned} \tag{3.16}$$

□

Finally, letting $G = F$ in (3.1), (3.9), (3.11), and (3.13) yields the formulas

$$\mathcal{F}_{\alpha,\beta}(F * F)_\alpha(y) = \left[\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\right]^2, \tag{3.17}$$

$$\delta(F * F)_\alpha(y|w) = \sqrt{2}\sum_{j=1}^{n}\langle\theta_j, w\rangle(F * F_j)_\alpha(y), \tag{3.18}$$

$$\left(\delta F(\cdot|w) * \delta F(\cdot|w)\right)_\alpha(y) = \sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j, w\rangle\langle\theta_l, w\rangle(F_j * F_l)_\alpha(y), \tag{3.19}$$

$$\left(\delta F(y|\cdot) * \delta F(y|\cdot)\right)_\alpha(w) = \frac{1}{2}\left[\delta F(y|w)\right]^2 - \frac{\alpha^2}{2}\sum_{j=1}^{n}\left[F_j(y)\right]^2 \tag{3.20}$$

for all y and w in K.

It is interesting to note that the left-hand side of each of the formulas (3.1), (3.2), (3.4), (3.7), (3.9), (3.11), (3.13), (3.17), (3.18), (3.19), and (3.20) involve exactly two of the operations of transform, convolution and first variation, while each right-hand side involves at most one of these three operations.

4. Relationships involving three concepts. In this section, we examine all of the various relationships involving the integral transform, the convolution product, and the first variation, where each concept is used exactly once. There are more than six possibilities since one can take the transform or the convolution with respect to either the first or the second argument of the variation. However, in view of formula (3.4) and (3.7), there are some repetitions. To exhaust all possibilities, we need to take the variation of the expressions in (3.1) and (3.2), the convolution of the expressions in (3.4) and (3.7), and the transform of the expressions in formulas (3.9), (3.11), and (3.13). It turns out that there are ten distinct formulas, and these are given by (4.1) through (4.10) below. We omit the details of the calculations used to obtain (4.1) through (4.10) because the techniques needed are similar to those used above in Sections 2 and 3.

Again, because of the theorems in Section 2, all of the functionals that arise in this section are automatically elements of E_0. As usual, F and G in E_0 are given by (1.7) and (1.12), respectively.

Formula 4.1. Taking the first variation of the expressions in (3.1) or taking the transform of the expressions in (3.9) with respect to the first argument of the variation and then using (2.23) and (2.24) yields the formula

$$\begin{aligned}
\delta\mathcal{F}_{\alpha,\beta}(F * G)_\alpha(y|w) &= \beta\mathcal{F}_{\alpha,\beta}\delta(F * G)_\alpha(\cdot|w)(y) \\
&= \mathcal{F}_{\alpha,\beta}F\left(\frac{y}{\sqrt{2}}\right)\delta\mathcal{F}_{\alpha,\beta}G\left(\frac{y}{\sqrt{2}}\,\middle|\,\frac{w}{\sqrt{2}}\right) \\
&\quad + \delta\mathcal{F}_{\alpha,\beta}F\left(\frac{y}{\sqrt{2}}\,\middle|\,\frac{w}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}G\left(\frac{y}{\sqrt{2}}\right) \\
&= \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y)\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y|w) \\
&\quad + \delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w)\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y)
\end{aligned} \tag{4.1}$$

for all y and w in K.

Formula 4.2. Taking the first variation of the expressions in (3.2) or replacing F with $\mathcal{F}_{\alpha,\beta}F$ and G with $\mathcal{F}_{\alpha,\beta}G$ in (3.9) yields the formula

$$\begin{aligned}&\delta(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_\alpha(y|w)\\&\quad=\frac{\beta}{\sqrt{2}}\sum_{j=1}^{n}\langle\theta_j,w\rangle\big[(\mathcal{F}_{\alpha,\beta}F_j*\mathcal{F}_{\alpha,\beta}G)_\alpha(y)+(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G_j)_\alpha(y)\big]\end{aligned}\tag{4.2}$$

for all y and w in K.

Formula 4.3. Taking the integral transform of the expressions in (3.9) with respect to the second argument of the variation yields the formula

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}\delta(F*G)_\alpha(y|\cdot)(w)&=\beta\delta(F*G)_\alpha(y|w)\\&=\frac{\beta}{\sqrt{2}}\sum_{j=1}^{n}\langle\theta_j,w\rangle\big[(F_j*G)_\alpha(y)+(F*G_j)_\alpha(y)\big]\end{aligned}\tag{4.3}$$

for all y and w in K.

Formula 4.4. Taking the integral transform of the expressions in (3.11) with respect to the first argument of the variation and then using (3.1) and (2.25) yields the formula

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)*\delta G(\cdot|w))_\alpha(y)&=\mathcal{F}_{\alpha,\beta}\delta F(\cdot|w)\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}\delta G(\cdot|w)\left(\frac{y}{\sqrt{2}}\right)\\&=\frac{2}{\beta^2}\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(y|w)\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(y|w)\end{aligned}\tag{4.4}$$

for all y and w in K.

Formula 4.5. Taking the integral transform of the expressions in (3.11) with respect to the second argument of the variation yields the formula

$$\begin{aligned}&\int_{C_0[0,T]}(\delta F(\cdot|\beta w+\alpha x)*\delta G(\cdot|\beta w+\alpha x))_\alpha(y)m(dx)\\&\quad=\beta^2(\delta F(\cdot|w)*\delta G(\cdot|w))_\alpha(y)+\alpha^2\sum_{j=1}^{n}(F_j*G_j)_\alpha(y)\end{aligned}\tag{4.5}$$

for all y and w in K.

Formula 4.6. Taking the integral transform of the expressions in (3.13) with respect to the first argument of the variation yields the formula

$$\begin{aligned}&\int_{C_0[0,T]}(\delta F(\beta y+\alpha x|\cdot)*\delta G(\beta y+\alpha x|\cdot))_\alpha(w)m(dx)\\&\quad=\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle\mathcal{F}_{\alpha,\beta}(F_jG_l)(y)-\frac{\alpha^2}{2}\sum_{j=1}^{n}\mathcal{F}_{\alpha,\beta}(F_jG_j)(y)\end{aligned}\tag{4.6}$$

for all y and w in K.

Formula 4.7. Taking the integral transform of the expressions in (3.13) with respect to the second argument of the variation yields the formula

$$\mathcal{F}_{\alpha,\beta}\big(\delta F(y|\cdot) * \delta G(y|\cdot)\big)_\alpha(w) = \frac{\beta^2}{2}\delta F(y \mid w)\delta G(y \mid w) \tag{4.7}$$

for all y and w in K.

Formula 4.8. Taking the convolution product of the expressions in (3.4) with respect to the first argument of the variation yields the formula

$$\begin{aligned}\big(\mathcal{F}_{\alpha,\beta}\delta F(\cdot|w) * \mathcal{F}_{\alpha,\beta}\delta G(\cdot|w)\big)_\alpha(y) &= \frac{1}{\beta^2}\big(\delta\mathcal{F}_{\alpha,\beta}F(\cdot|w) * \delta\mathcal{F}_{\alpha,\beta}G(\cdot|w)\big)_\alpha(y)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle\big(\mathcal{F}_{\alpha,\beta}F_j * \mathcal{F}_{\alpha,\beta}G_l\big)_\alpha(y)\end{aligned} \tag{4.8}$$

for all y and w in K.

Formula 4.9. Taking the convolution product of the expressions in (3.4) with respect to the second argument of the variation, or replacing F with $\mathcal{F}_{\alpha,\beta}F$ and G with $\mathcal{F}_{\alpha,\beta}G$ in (3.13) and using (2.26) yields the formula

$$\begin{aligned}&\big(\delta\mathcal{F}_{\alpha,\beta}F(y|\cdot) * \delta\mathcal{F}_{\alpha,\beta}G(y|\cdot)\big)_\alpha(w)\\ &\quad= \frac{1}{2}\delta\mathcal{F}_{\alpha,\beta}F(y|w)\delta\mathcal{F}_{\alpha,\beta}G(y|w) - \frac{\alpha^2}{2}\sum_{j=1}^{n}\big(\mathcal{F}_{\alpha,\beta}F\big)_j(y)\big(\mathcal{F}_{\alpha,\beta}G\big)_j(y)\\ &\quad= \frac{1}{2}\delta\mathcal{F}_{\alpha,\beta}F(y|w)\delta\mathcal{F}_{\alpha,\beta}G(y|w) - \frac{\alpha^2\beta^2}{2}\sum_{j=1}^{n}\mathcal{F}_{\alpha,\beta}F_j(y)\mathcal{F}_{\alpha,\beta}G_j(y)\end{aligned} \tag{4.9}$$

for all y and w in K.

Formula 4.10. Taking the convolution product of the expressions in formula (3.7) with respect to the second argument of the variation yields the formula

$$\begin{aligned}\big(\mathcal{F}_{\alpha,\beta}\delta F(y|\cdot) * \mathcal{F}_{\alpha,\beta}\delta G(y|\cdot)\big)_\alpha(w) &= \beta^2\big(\delta F(y|\cdot) * \delta G(y|\cdot)\big)_\alpha(w)\\ &= \frac{\beta^2}{2}\Bigg[\delta F(y|w)\delta G(y|w) - \alpha^2\sum_{j=1}^{n}F_j(y)G_j(y)\Bigg]\end{aligned} \tag{4.10}$$

for all y and w in K.

For completeness, note that taking the convolution product of the expressions in (3.7) with respect to the first argument of the variation, does not yield a new formula; we simply get formula (3.11) again.

Again it is interesting to note that the left-hand side of each of the formulas (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) involve all three of the operations of transform, convolution, and first variation, while each right-hand side involves at most two. Also note that formulas (3.1), (3.13), (4.4), (4.7), (4.9), and (4.10) are useful because they permit one to calculate $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y)$, $(\delta F(y|\cdot)*\delta G(y|\cdot))_\alpha(w)$, ..., and $(\mathcal{F}_{\alpha,\beta}\delta F(y|\cdot)*\mathcal{F}_{\alpha,\beta}\delta G(y|\cdot))_\alpha(w)$ without actually having to calculate the convolution products on the left-hand sides of formulas (3.1), (3.13), ..., and (4.10). It is usually harder to calculate convolution products than transforms and first variations.

5. Further results. It is well known, see for example [5, 10], that for all $F \in E_0$, all $y \in K$, and all a, b, and c in $\mathbb{C}$,

$$\begin{aligned}\int_{C_0[0,T]}\left(\int_{C_0[0,T]}F(aw+bx+cy)m(dw)\right)m(dx)\\ &= \int_{C_0[0,T]}F\left(\sqrt{a^2+b^2}z+cy\right)m(dz)\\ &= \int_{C_0[0,T]}\left(\int_{C_0[0,T]}F(aw+bx+cy)m(dx)\right)m(dw),\end{aligned}\tag{5.1}$$

and that

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{\alpha',\beta'}F)(y) = F(y) = \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)(y)\tag{5.2}$$

provided $\beta\beta' = 1$ and $\alpha^2 + (\beta\alpha')^2 = 0$. In particular, for all $y \in K$,

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{i\alpha/\beta,1/\beta}F)(y) = F(y) = \mathcal{F}_{i\alpha/\beta,1/\beta}(\mathcal{F}_{\alpha,\beta}F)(y)\tag{5.3}$$

for all nonzero complex numbers α and β.

If in (1.3) we replace α with $i\alpha/\beta$, then (5.3) enables us to express the convolution product of the transforms of F and G as a transform of the product of F with G.

Theorem 5.1. *Let α and β be nonzero complex numbers and let F and G be functionals from E_0 given by (1.7) and (1.12), respectively. Then for all $y \in K$,*

$$\begin{aligned}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y) &= \mathcal{F}_{\alpha,\beta}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y)\\ &= \mathcal{F}_{\alpha,\beta/\sqrt{2}}(FG)(y).\end{aligned}\tag{5.4}$$

Proof. Let $\alpha' = i\alpha/\beta$ and $\beta' = 1/\beta$. Using (3.1), it follows that the formula

$$\mathcal{F}_{\alpha',\beta'}(L_1*L_2)_{\alpha'}(y) = \mathcal{F}_{\alpha',\beta'}L_1\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\alpha',\beta'}L_2\left(\frac{y}{\sqrt{2}}\right)\tag{5.5}$$

holds for all L_1 and L_2 in E_0 and all $y \in K$. Letting $L_1 = \mathcal{F}_{\alpha,\beta}F$ and $L_2 = \mathcal{F}_{\alpha,\beta}G$ in (5.5) and then using (5.3) yields the formula

$$\begin{aligned}\mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{\alpha'}(y) &= \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}G)\left(\frac{y}{\sqrt{2}}\right)\\ &= F\left(\frac{y}{\sqrt{2}}\right)G\left(\frac{y}{\sqrt{2}}\right)\end{aligned}\tag{5.6}$$

for all $y \in K$. Next taking the integral transform $\mathcal{F}_{\alpha,\beta}$ of each side of (5.6) yields formula (5.4) as desired. □

Theorem 5.2. *Let α, β, F, and G be as in Theorem 5.1. Then for all y and w in K,*

$$\delta(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y|w)=\frac{\beta}{\sqrt{2}}\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)G(\cdot)+F(\cdot)\delta G(\cdot|w))\left(\frac{y}{\sqrt{2}}\right). \tag{5.7}$$

Proof. Using (5.4) and (2.25), we see that for all y and w in K,

$$\begin{aligned}\delta(\mathcal{F}_{\alpha,\beta}F*\mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y|w)&=\delta\mathcal{F}_{\alpha,\beta}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y|w)\\&=\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}(F(\cdot)G(\cdot))(y|w)\\&=\frac{\beta}{\sqrt{2}}\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)G(\cdot)+F(\cdot)\delta G(\cdot|w))\left(\frac{y}{\sqrt{2}}\right).\end{aligned} \tag{5.8}$$

□

Next, using (5.4), we obtain the following analogue of Formula 4.8.

Theorem 5.3. *Let F, G, α, and β be as in Theorem 5.1. Then for all y and w in K,*

$$\begin{aligned}&(\delta\mathcal{F}_{\alpha,\beta}F(\cdot|w)*\delta\mathcal{F}_{\alpha,\beta}G(\cdot|w))_{i\alpha/\beta}(y)\\&\quad=\beta^2\sum_{l=1}^{n}\sum_{j=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle\mathcal{F}_{\alpha,\beta/\sqrt{2}}(F_jG_l)(y).\end{aligned} \tag{5.9}$$

Proof. Using (3.4), (5.4), Theorem 2.3, and (2.23), we obtain

$$\begin{aligned}&(\delta\mathcal{F}_{\alpha,\beta}F(\cdot|w)*\delta\mathcal{F}_{\alpha,\beta}G(\cdot|w))_{i\alpha/\beta}(y)\\&\quad=\beta^2(\mathcal{F}_{\alpha,\beta}\delta F(\cdot|w)*\mathcal{F}_{\alpha,\beta}\delta G(\cdot|w))_{i\alpha/\beta}(y)\\&\quad=\beta^2\mathcal{F}_{\alpha,\beta}\left(\delta F\left(\frac{\cdot}{\sqrt{2}}\middle|w\right)\delta G\left(\frac{\cdot}{\sqrt{2}}\middle|w\right)\right)(y)\\&\quad=\beta^2\mathcal{F}_{\alpha,\beta}\left(\left[\sum_{j=1}^{n}\langle\theta_j,w\rangle F_j\left(\frac{\cdot}{\sqrt{2}}\right)\right]\left[\sum_{l=1}^{n}\langle\theta_l,w\rangle G_l\left(\frac{\cdot}{\sqrt{2}}\right)\right]\right)(y)\\&\quad=\beta^2\sum_{l=1}^{n}\sum_{j=1}^{n}\langle\theta_j,w\rangle\langle\theta_l,w\rangle\mathcal{F}_{\alpha,\beta/\sqrt{2}}(F_jG_l)(y)\end{aligned} \tag{5.10}$$

for all y and w in K. □

It is interesting to note that we can obtain analogues of Formulas 4.9 and 4.10 directly by use of (3.13) and (3.7) rather than using Theorem 5.1 as we did in Theorem 5.3 to obtain an analogue of Formula 4.8.

Theorem 5.4. *Let F, G, α, and β be as in Theorem 5.1. Then for all y and w in K,*

$$\begin{aligned}&(\delta\mathcal{F}_{\alpha,\beta}F(y|\cdot)*\delta\mathcal{F}_{\alpha,\beta}G(y|\cdot))_{i\alpha/\beta}(w)\\&\quad=\frac{1}{2}\delta\mathcal{F}_{\alpha,\beta}F(y|w)\delta\mathcal{F}_{\alpha,\beta}G(y|w)+\frac{\alpha^2}{2}\sum_{j=1}^{n}\mathcal{F}_{\alpha,\beta}F_j(y)\mathcal{F}_{\alpha,\beta}G_j(y),\\&(\mathcal{F}_{\alpha,\beta}\delta F(y|\cdot)*\mathcal{F}_{\alpha,\beta}\delta G(y|\cdot))_{i\alpha/\beta}(w)\\&\quad=\frac{\beta^2}{2}\delta F(y|w)\delta G(y|w)+\frac{\alpha^2}{2}\sum_{j=1}^{n}F_j(y)G_j(y).\end{aligned} \tag{5.11}$$

Example 5.5. Next, we briefly discuss the functionals $F(x) = \sum_{j=1}^{n}\langle\theta_j, x\rangle$, $G(x) = \exp\{F(x)\}$, $H(x) = F(x)\exp\{F(x)\}$, $M(x) = [F(x)]^2 = [\sum_{j=1}^{n}\langle\theta_j,x\rangle]^2$, and $N(x) = \sum_{j=1}^{n}[\langle\theta_j,x\rangle]^2$, all of which are elements of E_0. The following formulas follow quite readily for all y and w in K:

$$\mathcal{F}_{\alpha,\beta}F(y) = \beta F(y), \tag{5.12}$$

$$\delta F(y|w) = F(w), \tag{5.13}$$

$$\delta\mathcal{F}_{\alpha,\beta}F(y|w) = \beta F(w), \tag{5.14}$$

$$\mathcal{F}_{\alpha,\beta}G(y) = \exp\left\{\frac{n\alpha^2}{2} + \beta F(y)\right\}, \tag{5.15}$$

$$\delta G(y|w) = F(w)\exp\{F(y)\}, \tag{5.16}$$

$$\delta\mathcal{F}_{\alpha,\beta}G(y|w) = \beta F(w)\exp\left\{\frac{n\alpha^2}{2} + \beta F(y)\right\}, \tag{5.17}$$

$$\mathcal{F}_{\alpha,\beta}H(y) = [n\alpha^2 + \beta F(y)]\exp\left\{\frac{n\alpha^2}{2} + \beta F(y)\right\}, \tag{5.18}$$

$$\delta H(y|w) = [1 + F(y)]F(w)\exp\{F(y)\}, \tag{5.19}$$

$$\delta\mathcal{F}_{\alpha,\beta}H(y|w) = \beta F(w)[n\alpha^2 + \beta F(y) + 1]\exp\left\{\frac{n\alpha^2}{2} + \beta F(y)\right\}, \tag{5.20}$$

$$\mathcal{F}_{\alpha,\beta}M(y) = n\alpha^2 + [\beta F(y)]^2, \tag{5.21}$$

$$\delta M(y|w) = 2F(w)F(y), \tag{5.22}$$

$$\delta\mathcal{F}_{\alpha,\beta}M(y|w) = 2\beta^2 F(w)F(y), \tag{5.23}$$

$$\mathcal{F}_{\alpha,\beta}N(y) = n\alpha^2 + \beta^2 N(y), \tag{5.24}$$

$$\delta N(y|w) = 2\sum_{j=1}^{n}\langle\theta_j, w\rangle\langle\theta_j, y\rangle = \sum_{j=1}^{n}N_j(y)F_j(w), \tag{5.25}$$

$$\delta\mathcal{F}_{\alpha,\beta}N(y|w) = \beta^2\delta N(y|w) = \beta^2\sum_{j=1}^{n}N_j(y)F_j(w). \tag{5.26}$$

Finally, note that by using the various formulas in Sections 3 and 4 together with the formulas (5.12) through (5.26), we can immediately write down many additional formulas involving the specific functionals F, G, H, M, and N defined above in Example 5.5. For example, using (3.1), (5.15), and (5.21), we observe that

$$\mathcal{F}_{\alpha,\beta}(M * G)_\alpha(y) = \left[n\alpha^2 + \frac{\beta^2}{2}F^2(y)\right]\exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}}F(y)\right\}, \tag{5.27}$$

and hence using (5.13), (5.16), and (5.22),

$$\begin{aligned}\delta\mathcal{F}_{\alpha,\beta}(M * G)_\alpha(y|w) = {}&\left[n\alpha^2 + \frac{\beta^2}{2}F^2(y)\right]\frac{\beta}{\sqrt{2}}F(w)\exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}}F(y)\right\}\\ &+ \beta^2 F(y)F(w)\exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}}F(y)\right\}.\end{aligned} \tag{5.28}$$

Remark 5.6. For $\sigma \in [0,1)$, let E_σ be the space of all functionals $F : K \to \mathbb{C}$ of the form (1.7) for some positive integer n, where $f(\lambda_1,\ldots,\lambda_n)$ is an entire function of the

n complex variables $\lambda_1,\dots,\lambda_n$ such that

$$|f(\lambda_1,\dots,\lambda_n)| \le A_F \exp\left\{B_F \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\right\} \tag{5.29}$$

for some positive constants A_F and B_F. Note that if $\sigma = 0$, then $E_\sigma = E_0$ and for $0 < \sigma_1 < \sigma_2 < 1$, $E_0 \subsetneqq E_{\sigma_1} \subsetneqq E_{\sigma_2} \subsetneqq L_2(C_0[0,T])$.

A careful examination of the proofs of Theorems 2.1, 2.2, 2.3, and 2.4 shows that the conclusions of all four of these theorems hold for all F and G in E_σ, $0 \le \sigma < 1$. For example, to show that the conclusions of Theorem 2.1 hold for E_σ, let $F \in E_\sigma$ be given by (1.7) with f satisfying (5.29). Then proceeding as in the proof of Theorem 2.1, we obtain that $\mathcal{F}_{\alpha,\beta}F$ is given by (2.1) with h defined by (2.2) satisfying

$$|h(\lambda_1,\dots,\lambda_n)| \le A_{\mathcal{F}_{\alpha,\beta}F} \exp\left\{B_{\mathcal{F}_{\alpha,\beta}F} \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\right\} \tag{5.30}$$

with

$$A_{\mathcal{F}_{\alpha,\beta}F} = A_F\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left\{-\frac{u^2}{2} + B_F(2|\alpha\mu|)^{1+\sigma}\right\}du\right)^n < \infty, \tag{5.31}$$

and with $B_{\mathcal{F}_{\alpha,\beta}F} = B_F(2|\beta|)^{1+\sigma}$. Hence $\mathcal{F}_{\alpha,\beta}F$ exists and belongs to E_σ.

Some possible extensions. It seems likely that using the functionals in E_0 (or E_σ) as building blocks, one could show that the results established in this paper hold for larger classes of functionals.

For example, let $\{F_m\}_{m=1}^{\infty}$ be a sequence from E_0 such that $\lim_{m\to\infty} F_m(y)$ exists for all $y \in K$ and let $F(y) = \lim_{m\to\infty} F_m(y)$. Now the condition

$$|F_m(y)| \le A\exp\{B\|y\|_\infty\} \tag{5.32}$$

for all $y \in K$ and all $m = 1,2,\dots$ ensures the existence of the integral transform $\mathcal{F}_{\alpha,\beta}F$ since by the dominated convergence theorem,

$$\begin{aligned}
\lim_{m\to\infty} \mathcal{F}_{\alpha,\beta}F_m(y) &= \lim_{m\to\infty} \int_{C_0[0,T]} F_m(\alpha x + \beta y)m(dx) \\
&= \int_{C_0[0,T]} F(\alpha x + \beta y)m(dx) \\
&= \mathcal{F}_{\alpha,\beta}F(y)
\end{aligned} \tag{5.33}$$

for each $y \in K$. Example 5.7 shows that F need not belong to E_σ for any $\sigma \in [0,1)$.

It seems as though finding appropriate conditions to put on the sequences $\{F_m\}_{m=1}^{\infty}$ and $\{G_m\}_{m=1}^{\infty}$ from E_0 to ensure the existence of $(F * G)_\alpha$ should not be too difficult. However to proceed further, a major key would be to find appropriate conditions to put on the functionals $\{F_m\}_{m=1}^{\infty}$ in order to ensure the existence of δF.

Example 5.7. Let $\{\theta_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence of functions in $L_2[0,T]$, each of bounded variation on $[0,T]$. For $m = 1,2,\ldots$ and $y \in K$, let

$$F_m(y) = \exp\left\{\sum_{j=1}^{m} \frac{\langle\theta_j, y\rangle}{2^j C_j}\right\} \tag{5.34}$$

with C_j given by (1.6). Clearly $F_m \in E_0$ for each $m = 1,2,\ldots$.

Also for each $m = 1,2,\ldots$ and each $y \in K$,

$$|F_m(y)| \le \exp\left\{\|y\|_\infty \sum_{j=1}^{m} \frac{1}{2^j}\right\} \le \exp\{\|y\|_\infty\}. \tag{5.35}$$

But $\lim_{m\to\infty} F_m(y) = \exp\{\sum_{j=1}^{\infty}(\langle\theta_j, y\rangle/2^j C_j)\} \equiv F(y)$ is not an element of E_0 (or E_σ for $0 \le \sigma < 1$) because it depends upon $\langle\theta_m, y\rangle$ for every $m \in \{1,2,\ldots\}$ and so it cannot be written in the form (1.7) for any positive integer n; recall that $\{\theta_j\}_{j=1}^{\infty}$ is a complete orthonormal set of functions in $L_2[0,T]$.

Acknowledgment. The first author wishes to express his gratitude to Professors D. Skoug and G. Johnson for their encouragement and valuable advice as well as the hospitality of the University of Nebraska, Lincoln.

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